SUPPLEMENTAL MATERIALS

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Physical and Geometric Effects on the Classical Geodetic Observations in Small-Scale Control Networks

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Determination of deflection of vertical components using EGMs

The geoid height using normalized spherical harmonic coefficients is given by

$$N(\varphi,\lambda)=\frac{GM}{a\gamma}\sum_{n=2}^{n_{\max}}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n+1}\left(\Delta\bar{c}_{nm}\cos m\lambda+\bar{s}_{nm}\sin m\lambda\right)\bar{P}_{nm}(\sin\varphi) \tag{S1}$$

The DOV components, in the meridian and prime vertical directions, at any point of interest on the geoid surface, can be obtained by ((Heiskanen and Moritz 1967), p. 312):

$$\xi=-\frac{1}{R}\frac{\partial N}{\partial\varphi}\ , \tag{S2}$$

$$\eta=-\frac{1}{R\cos\varphi}\frac{\partial N}{\partial\lambda}, \tag{S3}$$

By inserting Eq. (S1) in Eqs. (S2) and (S3) the DOV components are obtained:

$$\xi(r,\varphi,\lambda)=\frac{GM}{ar\gamma}\sum_{n=2}^{n_{\max}}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n+1}\left(\Delta\bar{c}_{nm}\cos m\lambda+\bar{s}_{nm}\sin m\lambda\right)\left(\bar{P}_{nm+1}(\sin\varphi)-m\tan\varphi\bar{P}_{nm}(\sin\varphi)\right) \tag{S4}$$

$$\eta(r,\varphi,\lambda)=\frac{GM}{a\gamma r\cos\varphi}\sum_{n=2}^{n_{\max}}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n+1}m\left(\Delta\bar{c}_{nm}\sin m\lambda-\bar{s}_{nm}\cos m\lambda\right)\bar{P}_{nm}(\sin\varphi) \tag{S5}$$

where (r,φ,λ) are the geocentric distance, the latitude and longitude, GM is the product of the gravitational constant and Earth's mass, a is the semi-major axis of the reference ellipsoid, $\Delta\bar{c}_{nm}$ are the difference between the normalized geopotential coefficients and the harmonic coefficient generated by the normal gravity field, $\bar{s}_{nm}$ are the normalized geopotential coefficients, $\bar{P}_{nm}$ are the fully normalized Legendre polynomial of degree n and order m. The normal gravity γ can be computed at any point on the reference ellipsoid (e.g. GRS80) using Somigliana's formula (Moritz 1984). The geocentric distance (Sjöberg and Bagherbandi 2017, p. 36) can be obtained by:

$$\begin{aligned}X&=(\hat{N}+h)\cos\varphi\cos\lambda\\Y&=(\hat{N}+h)\cos\varphi\sin\lambda\\Z&=(\hat{N}(1-e^2)+h)\sin\varphi\end{aligned} \tag{S6}$$

where h is ellipsoidal height and $\widehat{N}$ is prime vertical.

$$\widehat{N} = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}} \tag{S7}$$

and finally, the geocentric distance is given by:

$$r = \sqrt{X^2 + Y^2 + Z^2} \tag{S8}$$

Evaluation of deflection of verticals using classic astronomical observations

To evaluate the obtained DOV components (SWEN17 model), one can compare them with those obtained from classic astronomical observations or advanced zenith camera systems (e.g. Hirt and Seeber 2002). We used the classic astronomical observations reported by Ekman and Ågren (2010) to perform this evaluation. They reanalyzed the coordinates of old fundamental observatories using astronomical, satellite positioning (GPS) and gravimetric DOV components in two old observatory towers in Stockholm (established 1748) and København (established in 1637). The Stockholm observatory data can be used to evaluate the obtained DOV components, similar to Ekman and Ågren (2010). The relation between the DOV components and the astronomic (Φ, Λ) and geodetic (φ, λ) latitudes and longitudes are given by ((Heiskanen and Moritz 1967), p. 187):

$$\begin{aligned} \xi^{astro-geo} &= \Phi - \varphi \\ \eta^{astro-geo} &= (\Lambda - \lambda)\cos\varphi \end{aligned} \tag{S9}$$

The geodetic latitude φ and longitude λ of the Stockholm observatory were obtained using GPS observations in The European Terrestrial Reference System 1989 (ETRS89). There are three campaigns for calculating the astronomical latitude (Φ) at the Stockholm observatory done by Wargentin (1759), Cronstrand (1811), and Selander (1835). The astronomical longitude (Λ) of the observatory was calculated using the difference between the local time of the Stockholm and Greenwich observatories. The local times are estimated from meridian transits of stars (see more details in Ekman and Ågren 2010). Two main campaigns were performed by Struve (1844) and Fuss and Nyren (1871) to calculate the longitude of the Stockholm Observatory, and the combined result of both campaigns is $\Lambda = 18^{\circ}03'29.8"$. Table S1 (in supplementary materials) summarizes the DOV components obtained by the SWEN17 model and the ones using Eq. (S7). The obtained quantities show that the difference between the SWEN17 model and the DOV obtained from the astronomical coordinates are 0.21 (using an average of $\xi^{astro-geo}$) and 0.08 arc seconds for ξ and η, respectively. Although the uncertainties of the astronomical latitude (Φ) obtained by different campaigns are not similar, all campaigns results are close to SWEN17 DOV results.

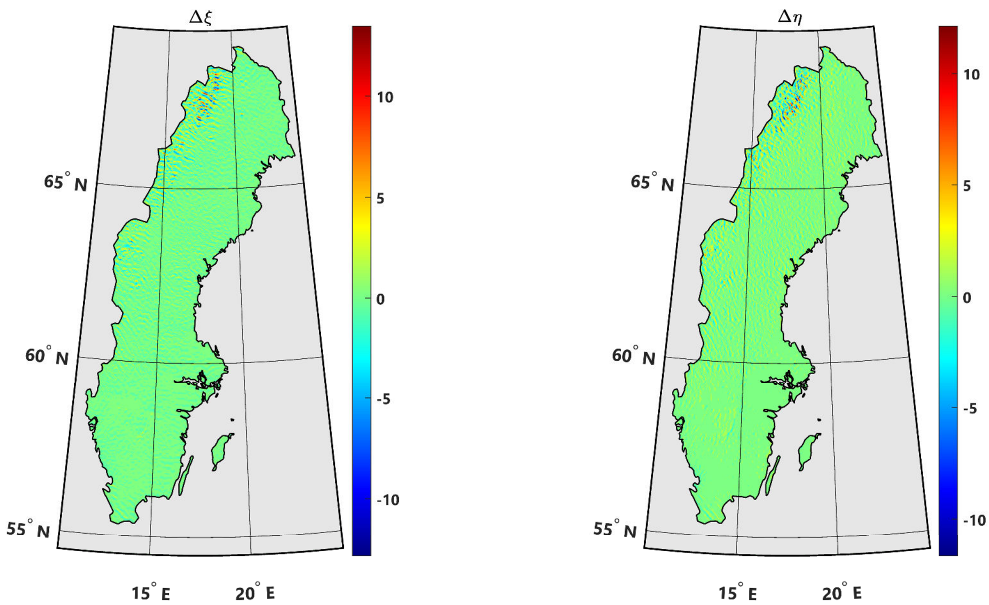

Fig. S1. Differences between the north-south (ξ) and east-west (η) components obtained by the SWEN17 and EGM2008 models in Sweden (The differences denoted by Δ in the figure). Unit: arc second.

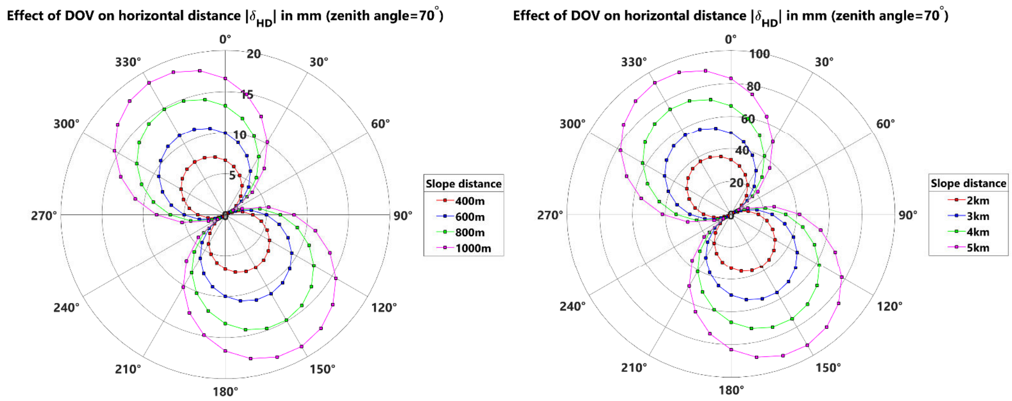

Fig. S2. Impact of the DOVs on the horizontal distances considering zenith/vertical angle equal to 70°, different baseline lengths, and azimuth angles using the SWEN17 model in **Kebnekaise** (the polar plots show max absolute value of $|\delta_{HD}|$). Unit: mm.

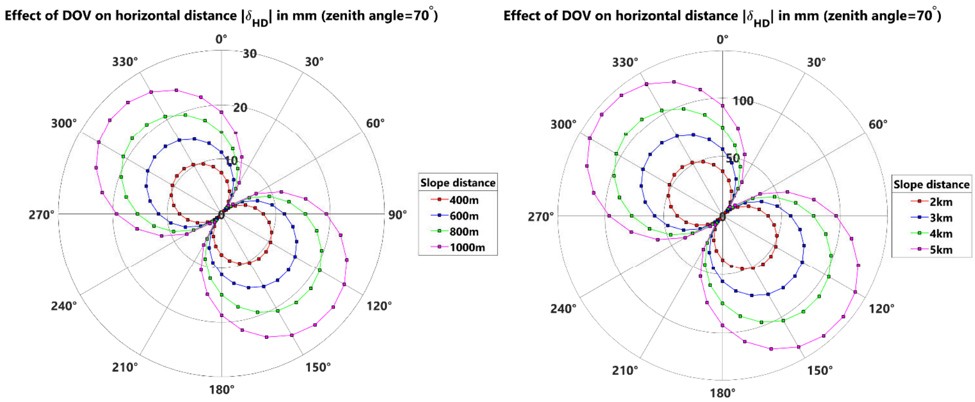

Fig. S3. Impact of the DOVs on the horizontal distances considering zenith/vertical angle equal to 70°, different baseline lengths, and azimuth angles using the SWEN17 model in **Umeå** (the polar plots show max absolute value of $\left|\delta_{HD}\right|$). Unit: mm.

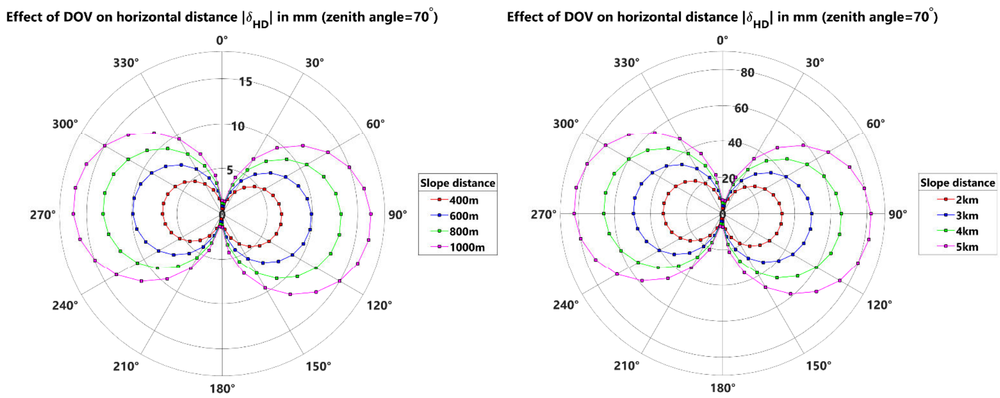

Fig. S4. Impact of the DOVs on the horizontal distances considering zenith/vertical angle equal to 70°, different baseline lengths, and azimuth angles using the SWEN17 model in **Mårtsbo** (the polar plots show max absolute value of $\left|\delta_{HD}\right|$). Unit: mm.

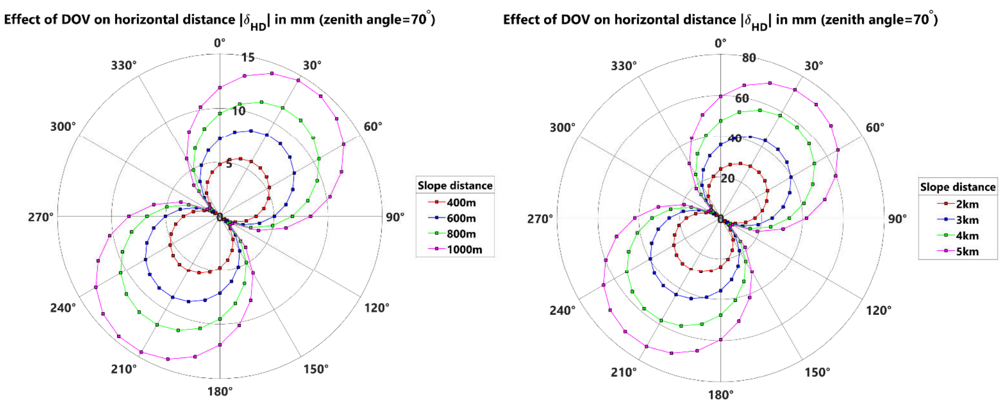

Fig. S5. Impact of the DOVs on the horizontal distances considering zenith/vertical angle equal to 70°, different baseline lengths, and azimuth angles using the SWEN17 model in **Skövde** (the polar plots show max absolute value of $\left|\delta_{HD}\right|$). Unit: mm.

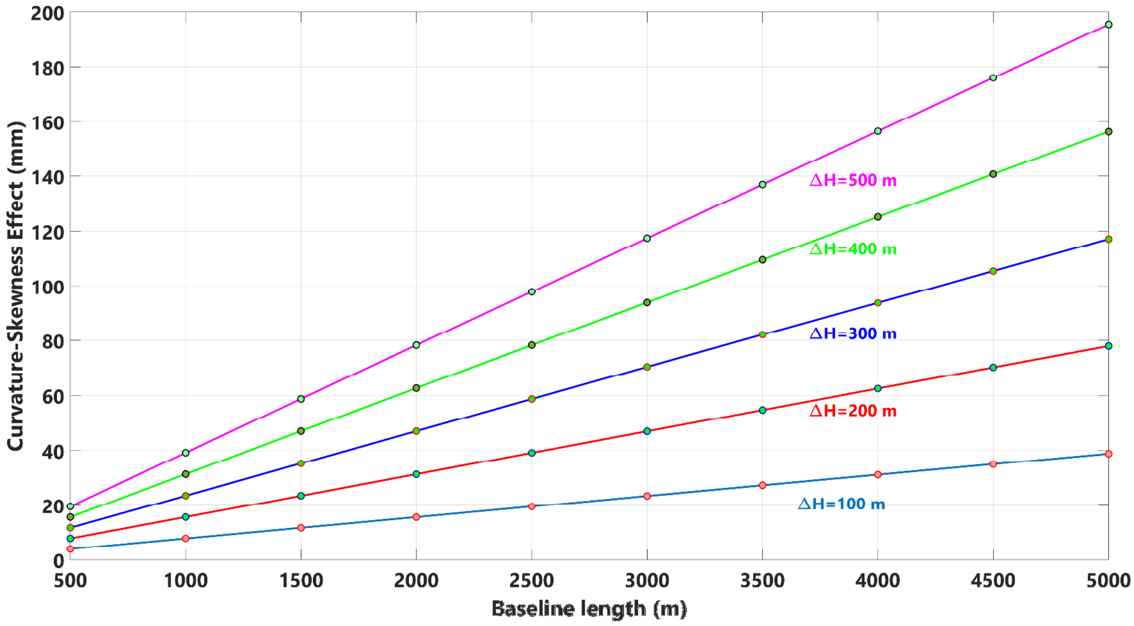

Fig. S6. The curvature-skewness effect on the slope distance reduction in reciprocal measurements for different baseline lengths and height differences.

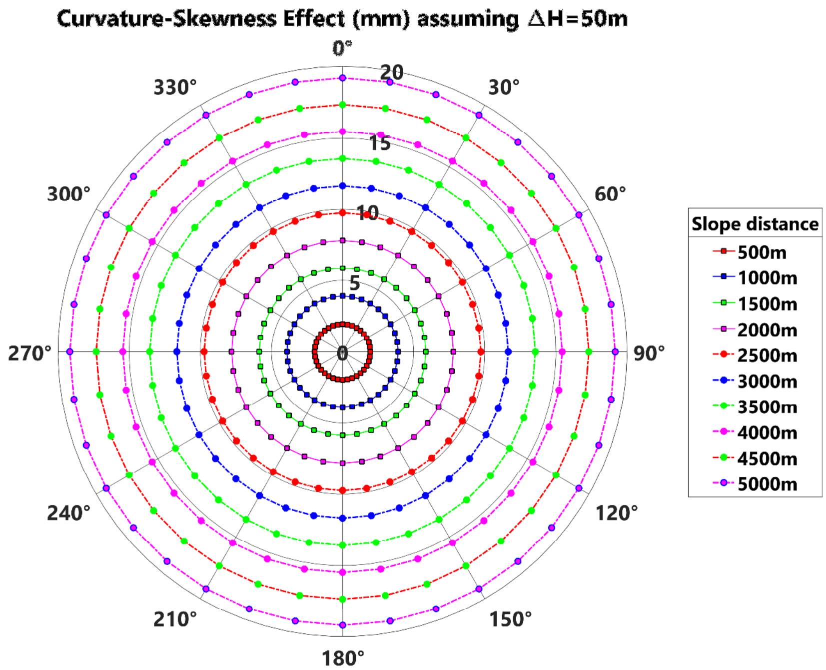

Fig. S7. Impact of curvature-skewness error on slope distance reductions assuming different azimuths (varying between 0° to 360° with 10° interval) and 50 m height difference.

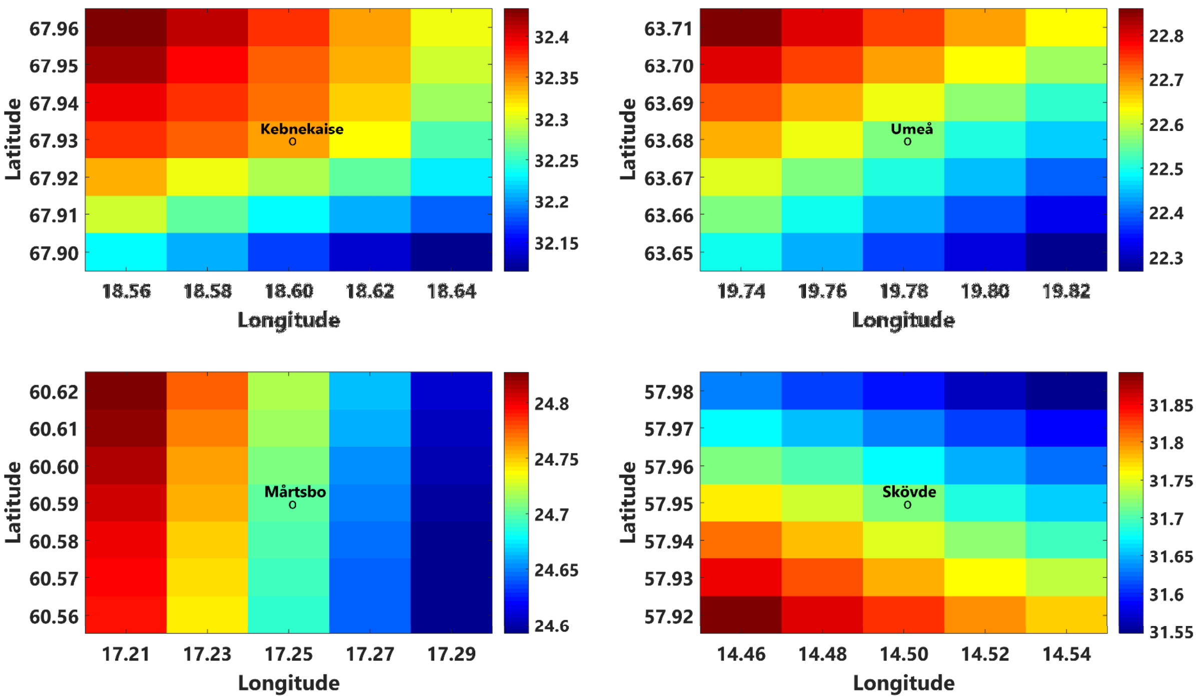

Fig. S8. The geoid height for the grid points at the test areas using the SWEN17 model. Unit: m.

Table S1: Comparison of the DOV component obtained from SWEN17 model and subtraction of astronomical and geodetic coordinates in Stockholm observatory ($\varphi = 59^{\circ}20'29.16''$, $\lambda = 18^{\circ}03'16.76''$). Unit: arc seconds

DOV component	Description	Magnitude
$\xi^{astro-geo}$	Using (Wargentin 1759) estimation for Φ $(59^{\circ}20'31.13'')$	1.97
	Using (Cronstrand 1811) estimation for Φ $(59^{\circ}20'34.8'')$	5.64
	Using (Selander 1835) estimation for Φ $(59^{\circ}20'33.8'')$	4.64
	Mean value of $\xi^{astro-geo}$	4.08
$\eta^{astro-geo}$		6.65
ξ_{SWEN17}		3.87
η_{SWEN17}		6.57

Table S2. Statistics of impact of the deflection of the vertical (DOV) on the horizontal distances (δ_{HD}) in **Kebnekaise** (the results are based on the variation of azimuth between 0°-360°).

Slope distance (m)	**Zenith angle = 70°** (mm)				**Zenith angle = 85°** (mm)			
	Max	Mean	Min	STD	Max	Mean	Min	STD
400	7.4	-0.2	-7.4	5.4	1.9	0.0	-1.9	1.4
600	11.1	-0.3	-11.1	8.1	2.8	-0.1	-2.8	2.1
800	14.9	-0.4	-14.9	10.7	3.8	-0.1	-3.8	2.7
1000	18.6	-0.4	-18.6	13.4	4.7	-0.1	-4.7	3.4
2000	37.1	-0.9	-37.1	26.9	9.5	-0.2	-9.5	6.8
3000	55.7	-1.3	-55.7	40.3	14.2	-0.3	-14.2	10.3
4000	74.3	-1.8	-74.3	53.7	18.9	-0.5	-18.9	13.7
5000	92.8	-2.2	-92.8	67.1	23.6	-0.6	-23.7	17.1

Table S3. Statistics of impact of the deflection of the vertical (DOV) on the horizontal distances (δ_{HD}) in **Umeå** (the results are based on the variation of azimuth between 0°-360°).

Slope distance (m)	**Zenith angle = 70°** (mm)				**Zenith angle = 85°** (mm)			
	Max	Mean	Min	STD	Max	Mean	Min	STD
400	10.7	-0.2	-10.7	7.7	2.7	-0.1	-2.7	2.0
600	16.1	-0.3	-16.1	11.5	4.1	-0.1	-4.1	2.9
800	21.4	-0.4	-21.4	15.4	5.5	-0.1	-5.5	3.9
1000	26.8	-0.5	-26.8	19.2	6.8	-0.1	-6.8	4.9
2000	53.6	-1.0	-53.6	38.5	13.6	-0.3	-13.7	9.8
3000	80.4	-1.5	-80.4	57.7	20.5	-0.4	-20.5	14.7
4000	107.2	-2.0	-107.2	77.0	27.3	-0.5	-27.3	19.6
5000	134.0	-2.5	-134.0	96.2	34.1	-0.7	-34.2	24.5

Table S4. Statistics of impact of the deflection of the vertical (DOV) on the horizontal distances (δ_{HD}) in **Mårtsbo** (the results are based on the variation of azimuth between 0°-360°).

Slope distance (m)	**Zenith angle = 70°** (mm)				**Zenith angle = 85°** (mm)			
	Max	Mean	Min	STD	Max	Mean	Min	STD
400	6.6	0.0	-6.6	4.7	1.7	0.0	-1.7	1.2
600	9.9	0.0	-9.9	7.0	2.5	0.0	-2.5	1.8
800	13.2	0.0	-13.2	9.4	3.4	0.0	-3.4	2.4
1000	16.5	0.0	-16.5	11.7	4.2	0.0	-4.2	3.0
2000	32.9	-0.1	-32.9	23.4	8.4	0.0	-8.4	6.0
3000	49.4	-0.1	-49.4	35.1	12.6	0.0	-12.6	8.9
4000	65.9	-0.2	-65.9	46.8	16.8	0.0	-16.8	11.9
5000	82.4	-0.2	-82.4	58.5	21.0	-0.1	-21.0	14.9

Table S5. Statistics of impact of the deflection of the vertical (DOV) on the horizontal distances (δ_{HD}) in **Skövde** (the results are based on the variation of azimuth between 0°-360°).

Slope distance (m)	**Zenith angle = 70°** (mm)				**Zenith angle = 85°** (mm)			
	Max	Mean	Min	STD	Max	Mean	Min	STD
400	5.8	0.1	-5.8	4.2	1.5	0.0	-1.5	1.1
600	8.7	0.2	-8.7	6.3	2.2	0.0	-2.2	1.6
800	11.6	0.3	-11.6	8.4	3.0	0.1	-3.0	2.1
1000	14.5	0.3	-14.5	10.5	3.7	0.1	-3.7	2.7
2000	29.0	0.6	-29.0	21.0	7.4	0.2	-7.4	5.3
3000	43.5	1.0	-43.5	31.5	11.1	0.2	-11.1	8.0
4000	58.1	1.3	-58.1	41.9	14.8	0.3	-14.8	10.7
5000	72.6	1.6	-72.6	52.4	18.5	0.4	-18.5	13.4

Table S6. Forward and backward DOV effect on the horizontal distances (δ_{HD}) considering zenith/vertical angle 85°, azimuth equal to 150°, and different baselines in Kebnekaise (Latitude: 67.93°, Longitude: 18.6°, Height: 1702.3m).

Distance	Latitude	Longitude	Height (m)	Forward (mm)	Backward (mm)	Difference (mm)
400	67.926884°	18.604787°	1667.7	1.9	1.9	0.0
600	67.925326°	18.607180°	1639.1	2.8	2.9	-0.1
800	67.923768°	18.609573°	1605.1	3.8	4.0	-0.2
1000	67.922210°	18.611965°	1567.2	4.7	5.1	-0.4
2000	67.914419°	18.623922°	1375	9.5	10.6	-1.1
3000	67.906628°	18.635871°	1305.7	14.2	17.9	-3.7
4000	67.898835°	18.647812°	1369.2	18.9	25.5	-6.6
5000	67.891042°	18.659745°	1410.2	23.6	32.8	-9.2

Table S7. Forward and backward DOV effect on the horizontal distances (δ_{HD}) considering zenith/vertical angle 85°, azimuth equal to 140°, and different baselines in Umeå (Latitude: 63.68°, Longitude: 19.78°, Height: 84.0 m).

Distance	Latitude	Longitude	Height (m)	Forward (mm)	Backward (mm)	Difference (mm)
400	63.677244°	19.785215°	74.144	2.7	2.7	0.0
600	63.675866°	19.787823°	69.419	4.0	4.1	0.0
800	63.674488°	19.790430°	64.889	5.4	5.5	-0.1
1000	63.673109°	19.793036°	60.605	6.7	6.8	-0.1
2000	63.666217°	19.806066°	44.325	13.5	13.5	0.0
3000	63.659324°	19.819090°	36.712	20.2	20.1	0.2
4000	63.652430°	19.832107°	34.081	26.9	26.4	0.6
5000	63.645535°	19.845118°	31.575	33.7	32.6	1.1

Table S8. Forward and backward DOV effect on the horizontal distances (δ_{HD}) considering zenith/vertical angle 85°, azimuth equal to 100°, and different baselines in Mårtsbo (Latitude: 60.595143°, Longitude: 17.258525°, Height: 32.0 m).

Distance	**Latitude**	**Longitude**	**Height** (m)	**Forward** (mm)	**Backward** (mm)	**Difference** (mm)
400	60.594518°	17.265741°	32.125	1.7	1.7	0.0
600	60.594205°	17.269349°	30.979	2.5	2.5	0.0
800	60.593893°	17.272957°	30.419	3.3	3.4	0.0
1000	60.593580°	17.276565°	29.871	4.2	4.2	0.0
2000	60.592014°	17.294604°	29.338	8.4	8.4	0.0
3000	60.590446°	17.312640°	26.915	12.5	12.2	0.3
4000	60.588876°	17.330675°	24.892	16.7	16.0	0.7
5000	60.587303°	17.348709°	23.065	20.9	20.0	0.9

Table S9. Forward and backward DOV effect on the horizontal distances (δ_{HD}) considering zenith/vertical angle 85°, azimuth equal to 30°, and different baselines in Skövde (Latitude: 57.95°, Longitude: 14.50°, Height: 262.0 m).

Distance	**Latitude**	**Longitude**	**Height** (m)	**Forward** (mm)	**Backward** (mm)	**Difference** (mm)
400	57.953116°	14.503390°	258.44	1.5	1.5	0.0
600	57.954674°	14.505086°	256.23	2.3	2.2	0.0
800	57.956231°	14.506781°	253.76	3.0	3.0	0.1
1000	57.957789°	14.508477°	251.05	3.8	3.7	0.1
2000	57.965578°	14.516957°	235.23	7.6	7.1	0.5
3000	57.973366°	14.525441°	219.23	11.3	10.4	0.9
4000	57.981153°	14.533929°	207.59	15.1	13.5	1.6
5000	57.988940°	14.542420°	203.12	18.9	16.7	2.2

Table S10. Computed geoid for the grid points at the test areas using the SWEN17 model.

Kebnekaise			Umeå			Skövde			Mårtsbo		
φ (degree)	λ (degree)	N (m)	φ (degree)	λ (degree)	N (m)	φ (degree)	λ (degree)	N (m)	φ (degree)	λ (degree)	N (m)
67.96	18.56	32.433	63.71	19.74	22.859	57.98	14.46	31.631	60.62514	17.21853	24.827
67.96	18.58	32.409	63.71	19.76	22.803	57.98	14.48	31.610	60.62514	17.23852	24.773
67.96	18.60	32.376	63.71	19.78	22.746	57.98	14.50	31.590	60.62514	17.25852	24.719
67.96	18.62	32.340	63.71	19.80	22.687	57.98	14.52	31.568	60.62514	17.27852	24.665
67.96	18.64	32.305	63.71	19.82	22.634	57.98	14.54	31.547	60.62514	17.29852	24.609
67.95	18.56	32.421	63.70	19.74	22.798	57.97	14.46	31.676	60.61514	17.21853	24.820
67.95	18.58	32.389	63.70	19.76	22.744	57.97	14.48	31.653	60.61514	17.23852	24.767
67.95	18.60	32.361	63.70	19.78	22.684	57.97	14.50	31.631	60.61514	17.25852	24.714
67.95	18.62	32.335	63.70	19.80	22.629	57.97	14.52	31.608	60.61514	17.27852	24.659
67.95	18.64	32.298	63.70	19.82	22.576	57.97	14.54	31.586	60.61514	17.29852	24.604
67.94	18.56	32.396	63.69	19.74	22.738	57.96	14.46	31.719	60.60514	17.21853	24.814
67.94	18.58	32.377	63.69	19.76	22.681	57.96	14.48	31.698	60.60514	17.23852	24.760
67.94	18.60	32.355	63.69	19.78	22.623	57.96	14.50	31.672	60.60514	17.25852	24.706
67.94	18.62	32.326	63.69	19.80	22.570	57.96	14.52	31.647	60.60514	17.27852	24.653
67.94	18.64	32.282	63.69	19.82	22.517	57.96	14.54	31.623	60.60514	17.29852	24.600
67.93	18.56	32.375	63.68	19.74	22.678	57.95	14.46	31.764	60.59514	17.21853	24.807
67.93	18.58	32.363	63.68	19.76	22.620	57.95	14.48	31.742	60.59514	17.23852	24.754
67.93	18.60	32.339	63.68	19.78	22.563	57.95	14.50	31.714	60.59514	17.25852	24.702
67.93	18.62	32.310	63.68	19.80	22.508	57.95	14.52	31.684	60.59514	17.27852	24.648
67.93	18.64	32.256	63.68	19.82	22.454	57.95	14.54	31.659	60.59514	17.29852	24.597
67.92	18.56	32.335	63.67	19.74	22.617	57.94	14.46	31.807	60.58514	17.21853	24.801
67.92	18.58	32.307	63.67	19.76	22.559	57.94	14.48	31.780	60.58514	17.23852	24.749
67.92	18.60	32.286	63.67	19.78	22.501	57.94	14.50	31.750	60.58514	17.25852	24.697
67.92	18.62	32.263	63.67	19.80	22.445	57.94	14.52	31.722	60.58514	17.27852	24.644
67.92	18.64	32.227	63.67	19.82	22.392	57.94	14.54	31.697	60.58514	17.29852	24.593
67.91	18.56	32.295	63.66	19.74	22.555	57.93	14.46	31.849	60.57514	17.21853	24.795
67.91	18.58	32.264	63.66	19.76	22.497	57.93	14.48	31.818	60.57514	17.23852	24.743
67.91	18.60	32.230	63.66	19.78	22.440	57.93	14.50	31.788	60.57514	17.25852	24.693
67.91	18.62	32.207	63.66	19.80	22.383	57.93	14.52	31.760	60.57514	17.27852	24.642
67.91	18.64	32.181	63.66	19.82	22.330	57.93	14.54	31.737	60.57514	17.29852	24.593
67.90	18.56	32.231	63.65	19.74	22.494	57.92	14.46	31.892	60.56514	17.21853	24.792
67.90	18.58	32.206	63.65	19.76	22.436	57.92	14.48	31.858	60.56514	17.23852	24.740
67.90	18.60	32.172	63.65	19.78	22.379	57.92	14.50	31.831	60.56514	17.25852	24.691
67.90	18.62	32.137	63.65	19.80	22.322	57.92	14.52	31.799	60.56514	17.27852	24.641
67.90	18.64	32.115	63.65	19.82	22.269	57.98	14.43	31.659	60.56514	17.29852	24.592

Table S11. The curvature-skewness angle in Umeå.

Baseline length (m)	Vertical skewness (d°, m′, s″)
0988.77	0, 0, 31.90
1490.18	0, 0, 48.11
1977.54	0, 1, 03.81
2439.03	0, 1, 18.77
2980.59	0, 1, 36.23
3487.45	0, 1, 52.63
3955.07	0, 2, 07.61
4541.38	0, 2, 26.58
5181.00	0, 2, 47.26

Table S12. The curvature-skewness angle in Skövde.

Baseline length (m)	Vertical skewness (d°, m′, s″)
1113.78	0, 0, 36.00
1625.16	0, 0, 52.48
2227.55	0, 1, 12.00
2522.50	0, 1, 21.50
3250.56	0, 1, 44.97
3544.84	0, 1, 54.55
4095.14	0, 2, 12.29
4455.10	0, 2, 23.99
5045.31	0, 2, 43.01

Table S13. The curvature-skewness angle in Mårtsbo.

Baseline length (m)	Vertical skewness (d°, m′, s″)
1094.34	0, 0, 35.31
1561.87	0, 0, 50.43
2189.22	0, 1, 10.64
2482.81	0, 1, 20.20
3124.37	0, 1, 40.89
3517.42	0, 1, 53.64
3996.34	0, 2, 09.08
4519.17	0, 2, 25.84
4589.46	0, 2, 28.28
4966.16	0, 2, 40.42

References:

Cronstrand, S. A. (1811). *Stockholms observatorii pol-högd bestämd.*

Ekman, M., & Ågren, J. (2010). *Reanalysing astronomical coordinates of old fundamental observatories using satellite positioning and deflections of the vertical*. Summer Institute for Historical Geophysics Åland Islands.

Fuss, V., & Nyren, M. (1871). *Bestimmung der Längen-Differenz zwischen den Sternwarten Stockholm und Helsingfors. Memoires de l'Academie Imperiale des Sciences.*

Heiskanen, W. A., & Moritz, H. (1967). Physical Geodesy. WH Freeman and Company. *San Francisco, CA*. ξ $^{-}$

Hirt, C., & Seeber, G. (2002). Astrogeodätische Lotabweichungsbestimmung mit dem digitalen Zenitkamerasystem TZK2-D. *ZfV - Zeitschrift Fuer Geodaesie, Geoinformation Und Landmanagement*, *127*(1), 388–396.

Moritz, H. (1984). Geodetic reference system 1980. *Bulletin Géodésique*, *58*(3), 388–398.

Selander, N. H. (1835). *Undersökning om Stockholms observatorii polhöjd.*

Sjöberg, L. E., & Bagherbandi, M. (2017). *Gravity inversion and integration*. Springer.

Struve, F. G. W. (1844). *Expedition chronometrique exceutee par ordre de sa majeste l'empereur Nicolas Ier entre Poulkova et Altona pour la determination de la longitude geographique relative de l'observatoire central de Russie.*

Wargentin, P. (1759). *Stockholms observatorii pol-högd bestämd.*