SUPPLEMENTAL MATERIALS

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Exact Winkler Solution for Laterally Loaded Piles in Inhomogeneous Soil

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Calculation of head stiffness for short piles

For short piles ($L < L_a$), the head stiffness terms are obtained using the boundary conditions at the pile head and pile tip. This can be carried out systematically using unitary shape functions $N_m(z)$, defined as follows:

$$y(z) = \boldsymbol{N}\boldsymbol{u_0} = \begin{bmatrix} N_1(z) & N_2(z) & N_3(z) & N_4(z) \end{bmatrix} \begin{bmatrix} y_0 \\ \theta_0 \\ y(L) \\ \theta(L) \end{bmatrix} \tag{S1}$$

where $\boldsymbol{N}$ and $\boldsymbol{u_0}$ are the vectors of shape functions and displacement/rotation boundary conditions, respectively. Each shape function can be obtained from the following expression:

$$\boldsymbol{N} = \begin{bmatrix} g_1(z) & g_2(z) & g_3(z) & g_4(z) \end{bmatrix} \begin{bmatrix} g_1(0) & g_2(0) & g_3(0) & g_4(0) \\ g_1'(0) & g_2'(0) & g_3'(0) & g_4'(0) \\ g_1(L) & g_2(L) & g_3(L) & g_4(L) \\ g_1'(L) & g_2'(L) & g_3'(L) & g_4'(L) \end{bmatrix}^{-1} \tag{S2}$$

where $g_m(z)$ are four independent solutions to the relevant governing equation.

Similarly to the case of a long pile case, a stiffness matrix, $\boldsymbol{K}^*$, can be obtained directly from the (exact) shape functions by differentiation. In this case, the matrix is 4 × 4 corresponding to the total degrees of freedom:

$$\boldsymbol{K}^* = \begin{bmatrix} K_{11}^* & K_{12}^* & K_{13}^* & K_{14}^* \\ K_{21}^* & K_{22}^* & K_{23}^* & K_{24}^* \\ K_{31}^* & K_{32}^* & K_{33}^* & K_{34}^* \\ K_{41}^* & K_{42}^* & K_{43}^* & K_{44}^* \end{bmatrix} = EI \begin{bmatrix} N_1'''(0) & N_2'''(0) & N_3'''(0) & N_4'''(0) \\ N_1''(0) & N_2''(0) & N_3''(0) & N_4''(0) \\ N_1'''(L) & N_2'''(L) & N_3'''(L) & N_4'''(L) \\ N_1''(L) & N_2''(L) & N_3''(L) & N_4''(L) \end{bmatrix} \tag{S3}$$

As only the pile head is of interest, $\boldsymbol{K}^*$ can be reduced to a 2 × 2 matrix, $\boldsymbol{K}$, using static condensation depending on the base boundary conditions. For fixed tip conditions, the third and fourth degrees of freedom of the pile are zero. Accordingly, the corresponding rows and columns of matrix $\boldsymbol{K}^*$ can simply be eliminated to give:

$$\boldsymbol{K} = \begin{bmatrix} K_{11}^* & K_{12}^* \\ K_{21}^* & K_{22}^* \end{bmatrix} \tag{S4}$$

For free tip conditions, the following expression is obtained:

$$\boldsymbol{K} = \begin{bmatrix} K_{11}^* & K_{12}^* \\ K_{21}^* & K_{22}^* \end{bmatrix} - \begin{bmatrix} K_{13}^* & K_{14}^* \\ K_{23}^* & K_{24}^* \end{bmatrix} \begin{bmatrix} K_{33}^* & K_{34}^* \\ K_{43}^* & K_{44}^* \end{bmatrix}^{-1} \begin{bmatrix} K_{31}^* & K_{32}^* \\ K_{41}^* & K_{42}^* \end{bmatrix} \tag{S5}$$

Similarly, for hinged tip conditions:

$$\boldsymbol{K} = \begin{bmatrix} K_{11}^* & K_{12}^* \\ K_{21}^* & K_{22}^* \end{bmatrix} - \begin{bmatrix} K_{14}^* \\ K_{24}^* \end{bmatrix} [K_{44}^*]^{-1} [K_{41}^* \quad K_{42}^*] \tag{S6}$$

Derivatives of Eqs. 11 and 26

The first three derivatives of the $h_m(z)$ solutions (Eq. 11 and 26) are provided in Eqs. S7-S10. Note that $\nu = 1/(n+4)$ (Eq. 26e) and therefore for the linear stiffness profile ($n = 1$), $\nu = 1/5$.

$$h_1'(z) = \lambda \frac{-(n+4)(\lambda z)^{n+3}}{(n+1)(n+2)(n+3)} {}_0F_3\left(; 2-\nu, 2-2\nu, 2-3\nu; -\nu^3(\lambda z)^{\frac{1}{\nu}}\right) \tag{S7a}$$

$$h_1''(z) = \lambda^2 \frac{-(n+4)(\lambda z)^{n+2}}{(n+1)(n+2)} {}_0F_3\left(; 1-\nu, 2-2\nu, 2-3\nu; -\nu^3(\lambda z)^{\frac{1}{\nu}}\right) \tag{S7b}$$

$$h_1'''(z) = \lambda^3 \frac{-(n+4)(\lambda z)^{n+1}}{(n+1)} {}_0F_3\left(; 1-\nu, 1-2\nu, 2-3\nu; -\nu^3(\lambda z)^{\frac{1}{\nu}}\right) \tag{S7c}$$

$$h_2'(z) = \lambda(\lambda z)^0 {}_0F_3\left(; 1-\nu, 1-2\nu, \nu; -\nu^3(\lambda z)^{\frac{1}{\nu}}\right) \tag{S8a}$$

$$h_2''(z) = \lambda^2 \frac{-(n+4)(\lambda z)^{n+3}}{(n+2)(n+3)} {}_0F_3\left(; 2-\nu, 2-2\nu, 1+\nu; -\nu^3(\lambda z)^{\frac{1}{\nu}}\right) \tag{S8b}$$

$$h_2'''(z) = \lambda^3 \frac{-(n+4)(\lambda z)^{n+2}}{(n+2)} {}_0F_3\left(; 1-\nu, 2-2\nu, 1+\nu; -\nu^3(\lambda z)^{\frac{1}{\nu}}\right) \tag{S8c}$$

$$h_3'(z) = \lambda(\lambda z)^1 {}_0F_3\left(; 1-\nu, 2\nu, 1+\nu; -\nu^3(\lambda z)^{\frac{1}{\nu}}\right) \tag{S9a}$$

$$h_3''(z) = \lambda^2(\lambda z)^0 {}_0F_3\left(; 1-\nu, 2\nu, \nu; -\nu^3(\lambda z)^{\frac{1}{\nu}}\right) \tag{S9b}$$

$$h_3'''(z) = \lambda^3 \frac{-(n+4)(\lambda z)^{n+3}}{2(n+3)} {}_0F_3\left(; 2-\nu, 1+2\nu, 1+\nu; -\nu^3(\lambda z)^{\frac{1}{\nu}}\right) \tag{S9c}$$

$$h_4'(z) = \lambda \frac{(\lambda z)^2}{2} {}_0F_3\left(; 3\nu, 1+2\nu, 1+\nu; -\nu^3(\lambda z)^{\frac{1}{\nu}}\right) \tag{S10a}$$

$$h_4''(z) = \lambda^2(\lambda z)^1 {}_0F_3\left(; 3\nu, 2\nu, 1+\nu; -\nu^3(\lambda z)^{\frac{1}{\nu}}\right) \tag{S10b}$$

$$h_4'''(z) = \lambda^3(\lambda z)^0 {}_0F_3\left(; 3\nu, 2\nu, \nu; -\nu^3(\lambda z)^{\frac{1}{\nu}}\right) \tag{S10c}$$